Matrix inverses

Def: If A is a square matrix, then a matrix B is an inverse of A if $AB = I$ and $BA = I$. If A has an inverse, then it is invertible.

Ex: $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

Then $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So B is an inverse of A, and A is an inverse of B.

Ex: Does $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ have an inverse? If $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an arbitrary matrix, then

$AB = \begin{bmatrix} c & d \\ 2c & 2d \end{bmatrix}$. But this is never I, since we can't have $c = 1$, $2c = 0$. Thus A is not invertible.

Is it possible for a matrix to have 2 different inverses?

Suppose a matrix A has inverses B and C. Then

$$B = IB = (CA)B = C(AB) = CI = C.$$

So $B = C$. That is, we've proven the following:

Theorem: If B and C are both inverses of A, then $B = C$.
That is, inverses are unique.

When A is invertible, the unique inverse is denoted A^{-1}.

Inverses of 2×2 matrices

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Def: The determinant of A is

$$\det A = ad - bc$$

The adjugate of A is the matrix

$$\operatorname{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We want to find conditions of A that make it invertible.

First notice the following:

$$A(\operatorname{adj} A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (\operatorname{adj} A) A$$

$ad - bc = \det A$, so as long as $\det A \neq 0$, we can divide through and get $A\left(\frac{1}{\det A} \operatorname{adj} A\right) = \frac{1}{\det A}\begin{bmatrix} \det A & 0 \\ 0 & \det A \end{bmatrix} = I$.

We summarize this in the following theorem:

Theorem: If A is a 2×2 matrix, then it is invertible if and only if $\det A \neq 0$. If $\det A \neq 0$, then its inverse is

$$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}\,A.$$

Later we will see how to give similar criteria for <u>any</u> square matrix.

Ex: $A = \begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix}$ has $\det A = 17$, so it's invertible,

and $$A^{-1} = \frac{1}{17}\begin{bmatrix} 7 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7/17 & -3/17 \\ 1/17 & 2/17 \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}$ Then $\det A = 0$, so A is not invertible.

Inverses and systems of equations

Recall that we can write a system of m equations in n variables as

$$\underbrace{A}_{\substack{m\times n \\ \text{coefficient} \\ \text{matrix}}}\ \underbrace{\vec{x}}_{\substack{\text{variable} \\ n\text{-vector} \\ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}} = \underbrace{\vec{b}}_{\substack{\text{constant} \\ m\text{-vector}}}$$

If A is an $n \times n$ invertible matrix (n equations + variables) then

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$\Rightarrow \quad I\vec{x} = A^{-1}\vec{b}$$
$$\Rightarrow \quad \vec{x} = \underbrace{A^{-1}\vec{b}}_{n\text{-vector}}$$

So $\vec{x} = A^{-1}\vec{b}$ is a solution to the system. Moreover, it's the <u>only</u> solution. We summarize this in a theorem:

Theorem: Let $A\vec{x} = \vec{b}$ be a system of n equations in n variables. If A is invertible, then the system has the unique solution $\vec{x} = A^{-1}\vec{b}$.

Ex: Consider the system

$$\begin{aligned} x + 2y &= 3 \\ -x + 4y &= 5 \end{aligned}$$

We can rewrite this as

$$\underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}}_{= A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$\det A = 4 + 2 = 6$, so A is invertible, and

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix},$$ so the system has unique solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 4/3 \end{bmatrix}$$

i.e. $x = 1/3$, $y = 4/3$.

Inverting $n \times n$ matrices

If A is an $n \times n$ invertible matrix, then we can go from $A \to I_n$ via a sequence of elementary row operations. It turns out, the same sequence of row operations takes $I_n \to A^{-1}$! That is, if we write A and I_n as blocks of a matrix, we have

$$[A \;\; I] \longrightarrow [I \;\; A^{-1}]$$

If A can't be brought to I, it is not invertible. If it can, it is.

We will see why this works in section 2.5, but for now we see how.

Ex: Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 0 & 4 & 2 \end{bmatrix}$.

$$[A \;\; I] = \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 4 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{②-2①} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 3 & -2 & 1 & 0 \\ 0 & 4 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}②}\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3/2 & 1 & -1/2 & 0 \\ 0 & 4 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{③-4②} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3/2 & 1 & -1/2 & 0 \\ 0 & 0 & 8 & -4 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{8}③}\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3/2 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & -1/2 & 1/4 & 1/8 \end{array}\right] \xrightarrow{②+\frac{3}{2}③} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/4 & -1/8 & 3/16 \\ 0 & 0 & 1 & -1/2 & 1/4 & 1/8 \end{array}\right]$$

$$\xrightarrow{①+③}\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1/2 & 1/4 & 1/8 \\ 0 & 1 & 0 & 1/4 & -1/8 & 3/16 \\ 0 & 0 & 1 & -1/2 & 1/4 & 1/8 \end{array}\right] \xrightarrow{①-②} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 3/8 & 5/16 \\ 0 & 1 & 0 & 1/4 & -1/8 & 3/16 \\ 0 & 0 & 1 & -1/2 & 1/4 & 1/8 \end{array}\right]$$

$$\underbrace{}_{I} \qquad \underbrace{}_{A^{-1}}$$

So $A^{-1} = \frac{1}{16}\begin{bmatrix} 4 & 6 & 5 \\ 4 & -2 & 3 \\ -8 & 4 & 2 \end{bmatrix}$

Some properties of invertible matrices

① If A is invertible, and $AB = AC$, then

$$A^{-1}AB = A^{-1}AC \Rightarrow B = C.$$ "left cancellation"

Right cancellation works too.

② If A is invertible, then we can show A^T is invertible too, and $(A^T)^{-1} = (A^{-1})^T$:

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

Similarly, $(A^{-1})^T A^T = I$, so $(A^{-1})^T$ is the inverse of A^T.

(3) If A and B are both invertible $n \times n$ matrices, then

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Similarly, $(B^{-1}A^{-1})(AB) = I$, so

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(4) $(A^{-1})^{-1} = A$

(5) If A is invertible, and $k \neq 0$ a real number, then

$$(kA)^{-1} = \frac{1}{k}A^{-1}.$$

We have already seen some connections between invertibility, systems of equations, and row operations. We summarize these connections in the following theorem, the most important theorem so far:

Theorem: Let A be an $n \times n$ matrix. The following five conditions are equivalent:

1. A is invertible.

$\boxed{2.}$ The system $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.

$\boxed{3.}$ $A \longrightarrow I_n$ via elementary row operations.

$\boxed{4.}$ $A\vec{x} = \vec{b}$ has at least one solution for every vector $\vec{b}$.

$\boxed{5.}$ There is an $n \times n$ matrix C such that $AC = I$.

Proof: We'll show that each of these implies the next and $5. \Rightarrow 1.$:

$\boxed{1.} \Rightarrow \boxed{2.}$: If A is invertible, then for any solution $\vec{x}$ of $A\vec{x} = \vec{0}$, we have
$$A^{-1}A\vec{x} = A^{-1}\vec{0}$$
$$\Rightarrow I\vec{x} = \vec{0}$$
$$\Rightarrow \vec{x} = \vec{0}$$
so the only solution is $\vec{x} = \vec{0}$.

$\boxed{2.} \Rightarrow \boxed{3.}$: If $\boxed{2.}$ is true, then A must have rank n, so we can go from $A \rightarrow I$ via row operations.

$\boxed{3.} \Rightarrow \boxed{4.}$: Take the augmented matrix $[A \mid \vec{b}]$.
Then $\boxed{3.}$ says we can go from $[A \mid \vec{b}] \rightarrow [I \mid \vec{c}]$ for some $\vec{c}$. Thus $\vec{x} = \vec{c}$ is a solution.

$\boxed{4.} \Rightarrow \boxed{5.}$: For each i, let $\vec{e}_i$ be the i^{th} column of

I. Then, by $\boxed{4.}$,

$A\vec{x} = \vec{e}_i$ has a solution, call it $\vec{c}_i$.

So $A\vec{c}_i = \vec{e}_i$. Take $C = [\vec{c}_1 \; \vec{c}_2 \dots \vec{c}_n]$ to be the matrix with the $\vec{c}_i$'s as columns. Then

$$\begin{aligned} AC &= A[\vec{c}_1 \; \vec{c}_2 \dots \vec{c}_n] \\ &= [A\vec{c}_1 \; A\vec{c}_2 \dots A\vec{c}_n] \\ &= [\vec{e}_1 \; \vec{e}_2 \dots \vec{e}_n] = I. \end{aligned}$$

$\boxed{5.} \Rightarrow \boxed{1.}$ Assume $AC = I$.

Then the system $C\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = I_n\vec{x} = AC\vec{x} = A\vec{0} = \vec{0}$.

So $\boxed{2.}$ holds for C. Thus, since $\boxed{2.} \Rightarrow \boxed{5.}$, there is a matrix C' such that $CC' = I$.

So $A = AI = A(CC') = (AC)C' = IC' = C'$.

So $AC = CA = I$, so A is invertible, and $C = A^{-1}$. $\square$

Note that this shows we only need to check inverses on one side. i.e. if $AC = I_n$ then A is invertible and $A^{-1} = C$.

Inverses of transformations

Let A be a square matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^n$ the induced transformation.

Question: If A is invertible, what does that tell us about T_A?

Consider $T_{A^{-1}}$, the transformation induced by A^{-1}. Then for any $\vec{x}$ in $\mathbb{R}^n$, we have

$$T_{A^{-1}}(T_A(\vec{x})) = A^{-1}(A\vec{x}) = I_n \vec{x} = \vec{x}, \text{ and } T_A(T_{A^{-1}}(\vec{x})) = \vec{x}.$$

That is, $T_{A^{-1}} \circ T_A = \underbrace{1_{\mathbb{R}^n}}_{\text{identity transformation}} = T_A \circ T_{A^{-1}}$

$T_{A^{-1}}$ is called an <u>inverse</u> function of T_A.

The converse holds as well (check this!). We summarize this as follows:

Theorem: If A is an $n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}$ the induced transformation, then A is invertible if and only if T has an inverse. In this case, $T^{-1} = T_{A^{-1}}$.

Practice problems: 2.4: 2ach, 3c, 4, 5cfh, 16